Mathematical Parables

Examples of mathematicians assisting each other and building upon instead of fearing their differences serve as modern parables for handling our own differences as scholars and friends.

J. W. Cannon

William P. Thurston, the best geometer in the world today, always wore plaid shirts and stretch denim jeans in his younger years. My children thus consider plaid shirts and stretch denim jeans standard costume for mathematicians. When Bill was to give the final talk of the International Congress of Mathematicians in Helsinki, we all wondered how he would dress for this formal occasion. Would he wear a plaid shirt and stretch denim jeans? Or would he succumb to the pressures of public appearance and dress in formal or semiformal attire? We watched from far back in the huge audience. He had indeed succumbed to societal pressures: he had ironed his plaid shirt.

A recent essay discusses academic dress and makes the following assertion: academic people dress with a formality inversely proportional to the confidence they have in their subject. Without doubt, said the essay, mathematicians are the worst dressed among all academics, since they have the most confidence in the reliability of their discipline. You can guess for yourself which departments were found to have the best-dressed professors. And tonight—well, you can guess from my black tuxedo and pleated white shirt just how much confidence I have in what I am going to say.

Elaine Sorensen congratulated me shortly after the Maeser Lecture was announced and said, "My only disappointment is that the committee still hasn't chosen a woman for the award." Her point was well taken. The Maeser Lecture is designed to honor an endeavor rather than a person. The lecturer represents a group of people who find academic life important and satisfying. The lecturers need to represent over time all of the large segments of this striving population, including women. The speakers are viewed as a little more important than they are because they represent not only the reality, but the aspirations of the population. Most of us can appreciate and hope for better things than we can actually accomplish.

Some would say that research and creative arts aren't that important in the eternal perspective. And in saying so they may be right. But this work, for me as a mathematician at least, is the best work that I can do, and I would be wrong to give it up for something that is even less important. It beats watching television by a country mile.

Some Generalizations about Mathematicians

Let us begin with a few general questions about mathematicians. These questions are representative of the ones I'm most often asked.

Regarding the traditional mathematics professor, George Pólya, a Hungarian mathematician, wrote:

> The traditional mathematics professor of the popular legend is absentminded. He usually appears in public with a lost umbrella in each hand. . . . He writes *a,* he says *b,* he means *c;* but it should be *d.* Some of his sayings are handed down from generation to generation. . . . [For example], "This principle is so perfectly general that no particular application of it is possible.
>
> Geometry is the art of correct reasoning on incorrect figures."
>
> After all, you can learn something from this traditional mathematics professor. Let us hope that the mathematics teacher from whom you cannot learn anything will not become traditional.[1]

How Does a Mathematician Work? Bill Floyd and his father are mathematicians. The elder Floyd was long-time provost of Thomas Jefferson's University of Virginia. Son Bill wanted to become a botanist, but in deference to his father, he applied to one graduate school in mathematics. Princeton accepted him. Commenting on the mathematician's thought process, Bill says, "R. H. Bing said he only worked on problems that you could think about while mowing the lawn. Any problem I've worked on I could think about while on a hike."[2]

Why Do People Become Mathematicians? G. H. Hardy wrote concerning the lure of mathematics:

> It is sometimes suggested, by lawyers or politicians or business men, that an academic career is one sought mainly by cautious and unambitious persons who care primarily for comfort and security. The reproach is quite misplaced. . . . [The mathematician] would have rejected their careers because of his ambition, because he would have scorned to be a man to be forgotten in twenty years.
>
> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas.*
>
> The mathematician's patterns, like the painter's or the poet's, must be *beautiful;* the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.[3]

Why Does Mathematics Exist? Science and mathematics exist because the universe is patterned. Mathematics is simply the study of these patterns, especially of their logical structure. Science decides whether a given pattern proposed by a mathematician does or does not approximate a specific aspect of reality.

People are amazed at the applicability of mathematics. For example, physicist Steven Weinberg writes about the application of linear algebra and matrix theory to Heisenberg's matrix mechanics, "This is one example of the spooky ability of mathematicians to anticipate structures that are relevant to the real world."[4] The mathematician would reply, "I wouldn't be a very good mathematician if I couldn't recognize fundamental patterns in the world around me."

Is Mathematics Worth Anything in the Real World? I always laugh inwardly when people ask, Is mathematics worth anything in the real world? Studying the patterns inherent in God's universe always seems more real to me than watching real-world television or engaging in real-world business. The mathematician wants to understand—to understand everything! The great German mathematician David Hilbert formulated this motto: "Wir müssen wissen. Wir werden wissen." "We must know! We will know!"[5] If mathematicians and scientists keep their eyes open for the richest and most significant patterns apparent about them, they can hardly avoid developing important results.

Consider three of the great scientific developments of this century: relativity theory, quantum mechanics, and the science of computing. Surely you will have a hard time finding three developments that have had a more profound effect on our world in this century. You may consider two of them as physics and the third as engineering, and they are. But each was founded in very large measure upon mathematics.

Einstein founded relativity theory on the mathematical principle of geometric invariance. The mathematics he used was the differential geometry of Gauss, Riemann, Poincaré, and Levi-Civita.

In his development of quantum mechanics, Erwin Schrödinger found exactly the mathematical methods he needed in the work of David Hilbert.[6] Concerning the discovery, Schrödinger wrote:

> It works!—the magnificent classical mathematics and the mathematics of Hilbert. These unfold everything so clearly before us, that all we have to do is to take it, without any labour and bothering; because the correct method is provided in time, as soon as one needs it, completely automatically.[7]

Mathematical logicians supplied the foundation for modern computing. Kurt Gödel, Emil Post, Steven Kleene, Barkely Rosser, Alan Turing, and others answered the question, What does it mean to calculate something? They found that all known methods of calculation can be reduced to procedures so simple that they can be implemented in a mechanical way. Based on this logic, another mathematician, John von Neumann, developed the idea of the stored computer program. With new developments in electronics, computers were on their way.

Do Mathematicians Delight in the Uselessness of Mathematics? Mathematicians have sometimes resisted being told by others what is real and what is important. The mathematician's point of view is well illustrated by several paintings of the Flemish painter Peter Bruegel the Elder. It would be helpful to refer to a collection of Bruegel's paintings, such as Marguerite Kay's book *Bruegel.*[8]

Ask yourself in each instance, What is the important issue at stake in the painting? As with mathematics, the important issue is not always obvious at first glance, and our experience may not make obvious what *is* important; therefore, we may have to be educated to realize that certain issues are critical.

First consider *The Numbering at Bethlehem.* The scene is a typical Flemish village, midwinter, with peasants walking or skating on a frozen river or lake, village children playing, and many people clustering around a rather prominent house in the lower-left foreground. Have you recognized the obvious fact that there is no room in this inn? Did you find Mary and Joseph and the donkey? Did you understand that one of the key moments of earthly existence is about to take place?

You won't have as much trouble with the next picture because you now know what to look for. The painting is entitled *The Procession to Calvary* and the scene is, of course, of Flemish peasants pursuing their everyday tasks. Far away on the hill stands a circle of people so distant as to be almost invisible. Upon close examination, you see that they are awaiting some special event. You are struck by the colorful red cloaks worn by a rather long procession of horseback riders. Almost invisible in the center of the painting is a man fallen upon his knees with a cross over his shoulder. The largest figure in the painting is beautiful: a woman in the lower-right corner who clearly sorrows and is being comforted by three companions—one male and two female.

In a third painting, *The Conversion of St. Paul,* you might have a little trouble identifying the important factors in the scene without some coaching from the artist. Business travelers in ancient costume with a Flemish flare are toiling up a dirt-covered mountain road. Amidst some very large trees, far away and somewhat in the background, many people surround a fallen traveler. One companion shades his eyes as he stares into the sky. Most of the travelers see nothing unusual or ignore what is happening. Would you, as an outsider, want to tell Paul what is really important at this moment? If he heard you at all, he might find your intrusion irritating.

Mathematicians care very much about the applications of their mathematics. As with Bruegel's work, the applications they have in mind, however, may have little to do with the ones you think they ought to have in mind.

How Do Mathematicians Choose What to Study? Much of mathematics is devoted to problems with an immense history. No field has a richer tradition. Most mathematical problems are too difficult to solve. Many mathematicians hack away at these old, difficult problems, bringing the difficulties into better focus, perhaps revealing new vistas and resolving old difficulties. But mathematicians are ready to consider any rich and interesting pattern, new or old.

What Is It Like to Be a Mathematician? My children say that only the children of mathematicians want to be mathematicians. I consider the life of a mathematician personally exciting. Mathematics has given me friends spread over the world and over time. I have spent days with the Greeks who lived hundreds of years before Christ—with Euclid, Eudoxus, Archimedes, Pythagoras, and Apollonius. I have spent months with Newton, with Euler, with the Bernoullis, with Gauss, Riemann, Poincaré, and Hilbert. I have personal friends on each continent and have traveled, because of mathematics, to four of the continents. I have personal mathematical friends in almost every state of the Union and in many foreign countries. They have invariably treated me with kindness and consideration and respect.

Let me give you an idea of the life of mathematics by outlining in the next section a problem in which I have been involved. It is not the most important problem in mathematics—only the best I can do. The account will be simplified. A reporter suggested to physicist Richard Feynman that he might respond to the question, "What did you win the [Nobel] Prize for?" with "Listen, buddy, if I could tell you in a minute what I did, it wouldn't be worth the Nobel Prize."[9] Of course, I haven't won the Nobel Prize. Mathematicians don't have a Nobel Prize. They have a Fields Medal.[10] And I haven't won the Fields Medal, and won't, since it is given only to mathematicians under the age of forty. As I discuss some general goals, ideas, and events, you may be able to gain a feeling for what it is like to be a mathematician.

Mathematical Parallels, or Parabolas: The Double Suspension Problem

Both *parable* and *parabola* mean to set beside or in parallel. Our mathematical parallels are called parabolas while our people parallels are called parables. I will start with parabolas. The parabolas I consider are geometry versus algebra and large versus small.

Geometry versus Algebra. Our models of space are built on line and number, geometry and algebra, respectively. In ancient Greece, the Pythagoreans viewed the entire universe as built upon number. They imagined the line as a succession of indivisible objects that could be counted, like beads on a string. When Hippasus, one of their colleagues, proved that this view was wrong, they were so shocked they drowned him in the sea.[11] Being a mathematician is sometimes dangerous.

History has since modified our vision of line and number. For millennia, the number zero was not recognized as a number—it was a nothing. In the first century after Christ, someone introduced zero as a placeholder. Arithmetic as we know it, however, was not widely accepted until A.D. 1300. Finger arithmetic was standard fare in books as recently as A.D. 1500.[12]

Beginning with the eighteenth century, the concepts of line and real number were fused, setting the stage for a unified geometric-arithmetic model for space. A number became simply a point in the line, and the line could be viewed as a varying number. The line was the basis for one-variable mathematics.

One variable, however, is not enough to measure a complex world. Are three space variables enough? Does it make any sense to talk about space with more than three or four variables?

Mathematician Mary Ellen Rudin once wrote in a letter:

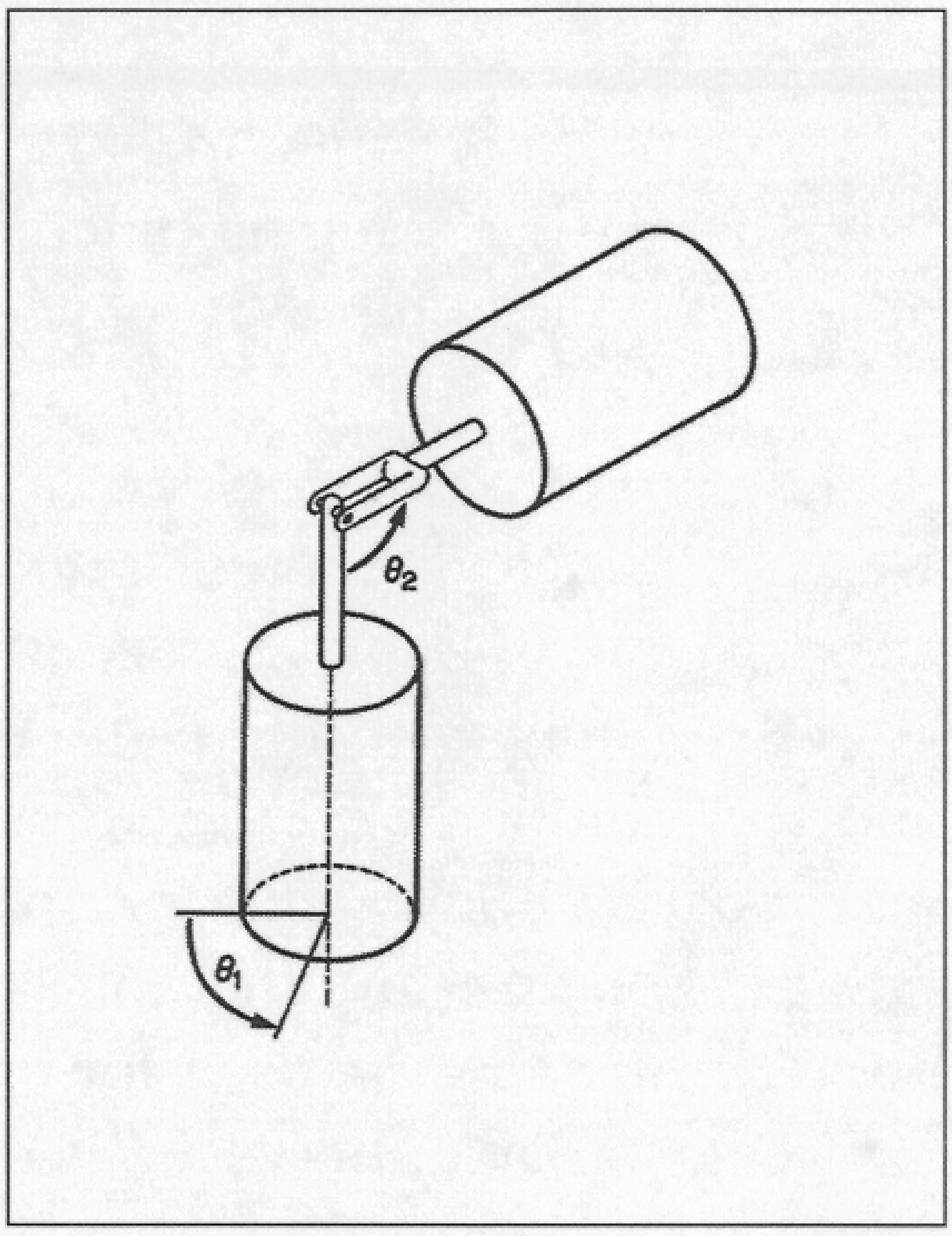
> My one total disaster [as a speaker] was a talk that I gave to the winners of Presidential awards for high school science teaching. It was at the State Department. The law forbade me to have any visual aid. I could only wave my hands. I'm not sure a single non-Ph.D. mathematician in the room understood anything I said. I got exactly one question, asked by perhaps 15 different people: "How can one possibly have more than 4 dimensions?"[13]

Mary Ellen Rudin grew up in a small Texas town. She wrote more than seventy research papers while tending her small children in her very open home designed by Frank Lloyd Wright. She covered the dangerous gaps in the open stairway with fishnet until the children were old enough to be safe from falling.

I should at this point ask a mother with six children (Mary Ellen had only four) to explain how many variables complicate her life. If she can get by with three variables, she has either lost four children or three children and their father.

Life is high dimensional. The mathematics of life is equally high dimensional. As one might say, the complications of life are manifold; *manifold* is the name given to a mathematical model with more than one variable. The mathematical term *manifold* should be learned by every educated person.

A robot arm formed from links illustrates the term *manifold.* The position of each link, relative to the previous link, is given by one variable. Each link raises the mathematical description of the arm by one dimension. Robot arms with six links have been built. The possible positions of such an arm require six variables for their descriptions and therefore form a manifold of six dimensions.



A robot arm. This robot arm is an example of the high dimensionality of life. The position of each link must be defined by a variable of one dimension. Because this arm has two free joints, a specific position of the arm must be described in terms of two dimensions.

Helaman Rolfe Pratt Ferguson is a noted Mormon mathematical artist who believes in the power of high-dimensional art. He asks the question, "Why does a sculpture affect us more when we see it in person than when we see a picture?" His answer? "The physical viewing of sculpture is a high-dimensional event." Ferguson comments:

> In examining sculpture, one should use all of the body, not just the eyes. Our feet supply a dimension that brings us in proximity to the object. In order to feel the object we bring into play the joints of feet, ankles, knees, waist, neck, shoulders, elbows, wrists, and fingers. The nerve impulses from fingertips and palms add to the sensory perception. With truly binocular vision at play with the twelve ocular muscles, with all of the muscles and joints and nerves sending their separate messages to our brains, when we view and feel the object at close range, we get a huge-dimensional view of a physical sculpture.[14]

Manifolds can be geometrically beautiful. They can be curved. They are generally very high dimensional. The notion of a manifold is much more basic to physics and mathematics than the better-known notions of a black hole, quasar, neutrino, or big bang. (This last point reveals the difference between mathematicians and physicists: physicists are better at advertising their work. If mathematicians solving some equations discover a singularity in the solution, they report, "There is a singularity in the solution of these equations." On the other hand, if physicists discover the same singularity by the same methods, they say, "The universe was created with a Big Bang [read, singularity in an equation]. Give us a billion dollars to study it.")

From 1821 until 1848, Karl Friedrich Gauss[15] conducted extensive geodetic surveys. (Geodesy is a branch of mathematics involving features of the earth, like surface, shape, size, and gravity.) These surveys suggested problems connected with curved surfaces. The resulting studies began the mathematics of relativity and the geometric study of manifolds.[16]

Gauss was followed by Bernhard Riemann, born in Germany in 1826, the son of a Lutheran pastor. In 1857, at the age of thirty-one, Riemann became an assistant professor. On his minuscule salary, he supported himself and three sisters. Riemann married at the age of thirty-six, but he contracted tuberculosis and died at the age of thirty-nine: "He said to [his wife], 'Kiss our child.' She repeated the Lord's prayer with him; . . . at the words 'Forgive us our trespasses' he looked up devoutly; she felt his hand grow colder in hers. He served his God faithfully, as his father had, but in a different way."[17]

Riemann revolutionized everything he touched. With Gauss, he fathered the geometry that led to relativity theory.[18] Here is how Riemann explained the notion of manifold to the general audience:

> If one travels in a continuous manner from one position to another, then the intermediate points through which one travels form a one-fold manifold. We then think of this entire one-fold manifold sliding from its given position over into another completely different position so that every point of the first passes into a specific point of the second. The simple manifold, by its motion, sweeps out what we call a two-fold extended manifold. And it is easy to imagine continuing this construction into arbitrarily many dimensions.[19]

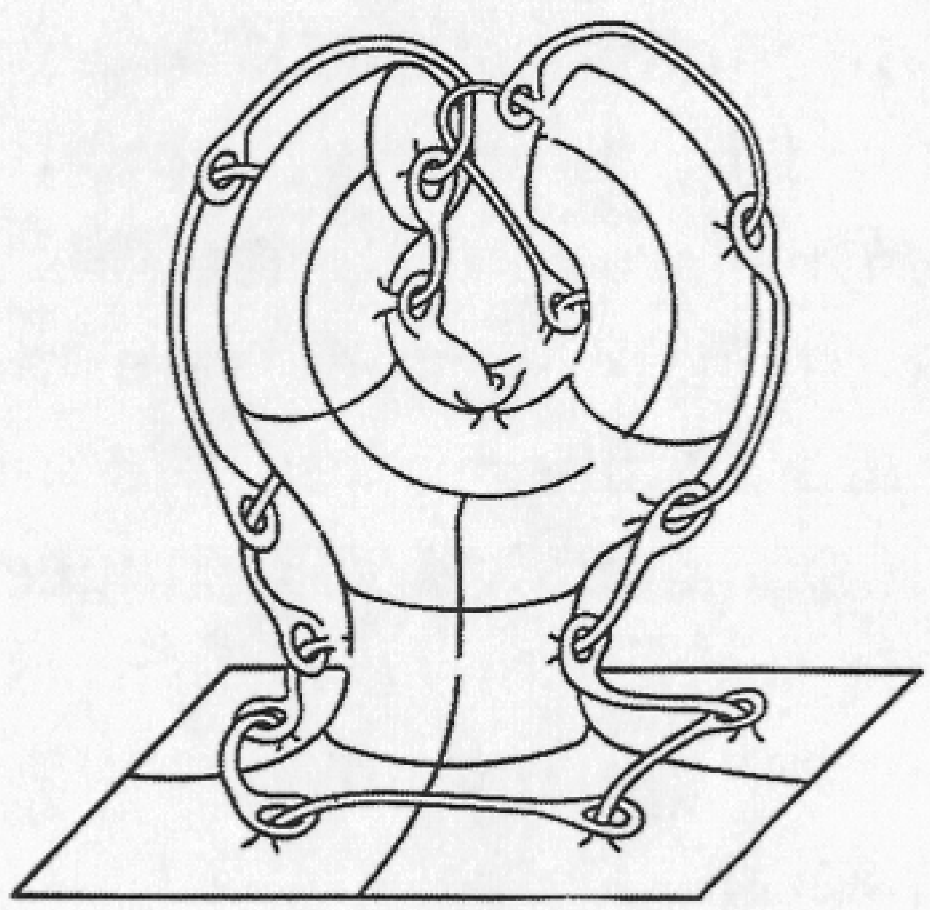
At the turn of the century, Henri Poincaré was the world's greatest mathematician. With H. A. Lorentz and Albert Einstein, he discovered the theory of special relativity, based on the theory of curved manifolds. Poincaré studied the properties of manifolds by means of a new method that we now call topology. (I consider myself a topologist.) Poincaré explained: "Topology allows us to recognize qualitative relations in spaces of more than three dimensions. Topology renders service analogous to that rendered in low dimensions by pictures."[20]

In summary, the most common mathematical model of space is the manifold, based on many variables, each variable being a number that as it varies, sweeps out a constituent direction or line in the manifold. This fusion of line, number, manifold, and space completes our first mathematical parallel or parable. Our second parabola is more modern, more specialized. It concerns the topology of manifolds. Here we are concerned with the conflict between the very large and very small.

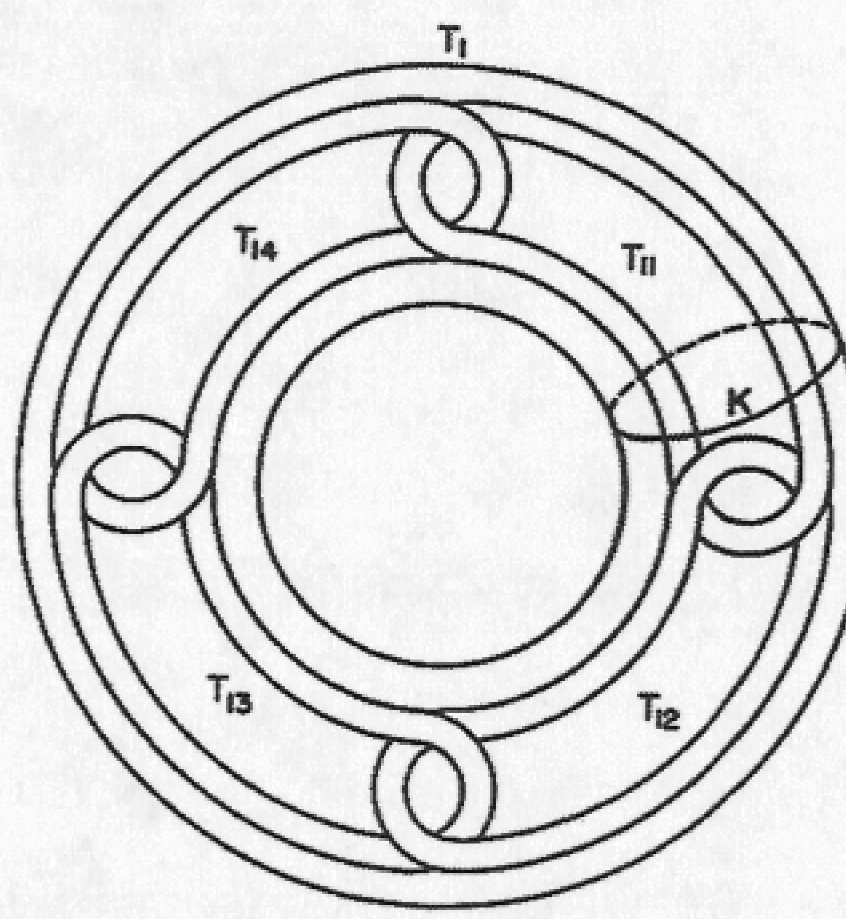
Large versus Small. People have always disagreed about the most important things to study. For example, are big things more important, or are little things? Relativity theory studies big things; quantum theory, small things. Manifolds are a good model for relativity theory, a poor model for quantum theory. Hence, two schools of thought have developed. One says only the large-scale structure of manifolds is important. The second says anything that happens in the small happens in the large and vice versa. Both small and large are important. The theories of small and large developed independently according to the thought and taste of the participants. An advocate of the large view was John Milnor of Princeton University. An advocate of the small view was R. H. Bing from Texas.

In an undergraduate class at Princeton University, freshman John Milnor solved within a week the problem his professor posed as an illustration of an important unsolved problem in mathematics.[21] Milnor went on to a brilliant career and became a Fields Medalist.[22] He claimed we should study high dimensions algebraically and prove theorems about infinitely many dimensions at once.

R. H. Bing, on the other hand, was a high-school football coach from Texas. The stereotypical complaint of mathematicians about high-school education is that the football coach is asked to teach mathematics classes although the stereotypical football coach scores lower than his students on standardized exams. Bing, however, was a football coach who went on to become president of both the American Mathematical Society and the Mathematical Association of America and was elected to the National Academy of Sciences. He never lost his love for football, once paying $700 for tickets to the Texas–Oklahoma football game. He loved to show just how surprising things could be, and he gave wonderful names to his examples. Thanks to him we now talk of the "Bing Sling," the "Dogbone Space," the "Hooked Rug," "Crumpled Cubes," and many more. In a deep Texas accent, he spoke of "epslums and deltas," of "baseball moves" and "pill boxes."[23] Bing's central claim was that we should really understand low dimensions with all of their warts. We should understand geometry geometrically and develop specific examples.



Hooked Rug



Bing Sling

We will call the two schools of thought developed by Milnor and Bing the Princeton School and the Texas School. The Princeton School approached things from an algebraic and global point of view. The Texas School approached things from a geometric and local point of view. The Princeton School felt that they had developed the algebra to the point where they almost had the whole story figured out. The Texas School developed complicated counter examples showing that things weren't as simple as the

Princeton School supposed. The Princeton School discounted the examples because the Texas School used limiting arguments that made sense only in the small.

Then the Princeton School ran up against a paradox. The 5-sphere is a well-known manifold that can be pieced together from eight 5-dimensional diamonds. The Princeton School, however, discovered another important 5-dimensional space that could be pieced together from twelve 5-dimensional diamonds. The new space satisfied all of their global prejudices. By their algebraic methods, the new space was indistinguishable from the 5-sphere. If the new space were a manifold, it could be only the 5-sphere. However, geometrically it obviously was not the 5-sphere because the only possible way it could be the 5-sphere would be that four edges of the twelve diamonds were "infinitely knotted" in the space. But how can a straight edge of a flat polyhedral space be infinitely knotted in that flat space?

The Princeton School was caught in a quandary. Either this space was not a manifold and their algebraic methods were inadequate to the task of understanding manifolds, or this space was a manifold and polyhedral pieces could be infinitely knotted, an absurd idea. The issue at stake was this: are global methods adequate to study things in the large, or must one study vanishingly small things to understand large things? Milnor asked the inconceivable: is it possible this space is a manifold?[24] This question became known as the famous double suspension problem. Is the double suspension of a homology sphere a manifold?

One of my very good instructors at the University of Utah, Les Glaser, soon proposed a proof showing that the space was not a manifold.[25] The idea of the proof was really clever. It used local limiting techniques inspired by Bing's work. Unfortunately, the proof was incorrect.[26] Making mistakes in mathematics is so easy. Everyone makes them. You try to catch your own. If you don't, someone else catches them for you. Slowly, the difficulties are understood and ironed out.

My story now leaves the question of mathematical taste in the United States during the 1960s and moves to Russia. The account here is most complicated. I leave it purposely confusing because I originally found it so myself.

M. A. Štan'ko. Štan'ko, a Russian mathematician, provided one of the crucial ideas used to solve the double suspension problem. 1971. Courtesy Robert J. Daverman.

M. A. Štan'ko, a young Russian mathematician, decided to see whether it is always possible to unknot an infinitely knotted object that is in high-dimensional space.[27] His idea was this: look at a three-dimensional slice of that high-dimensional space. The part of the space that sticks into that slice is knotted in ways that we understand. If we aren't too fussy, we can unknot a knotted set in three-dimensional space by simply cutting apart any knotting that we see, rearranging the strands so they no longer look knotted, and fusing (taping?) them back together in an unknotted position. Alternatively, we can think of one strand as being physically transparent to any other strand so that the strands can simply be unknotted by pulling them through one another.

But what are we to do with the part of the space that is all around the three-dimensional slice that we have chosen? We have to move the parts of the space in nearby slices as well. And in the nearby slices we do not have the same control that we had in the slice that we were viewing. Things that weren't tangled become tangled, and strands that had missed one another hit one another. It is one big mess!

Štan'ko noticed that, if he were to look ahead, he could prepare for the problem in advance by first moving some things out of the way in nearby slices. But moving things out of the way in nearby slices creates the same type of unknotting problem found in the first problem. But that difficulty can again be solved by looking ahead and pushing things out of the way of the things that need to be moved out of the way. But this third move can be prepared for by a still earlier fourth move, which can be prepared for by a still earlier fifth move, and so forth. That is, we only have to look ahead infinitely many steps. We do these infinitely many steps, infinitely often, in infinitely many carefully chosen places, and the unknotting problem is solved!

Did you follow Štan'ko's argument? Neither did I. I stared at the critical four pages of his paper[28] every day for a month. His idea, whatever it was, seemed really important, and I was determined to understand it. And then all at once I saw how simple it was. Trivial! as they say in mathematics.

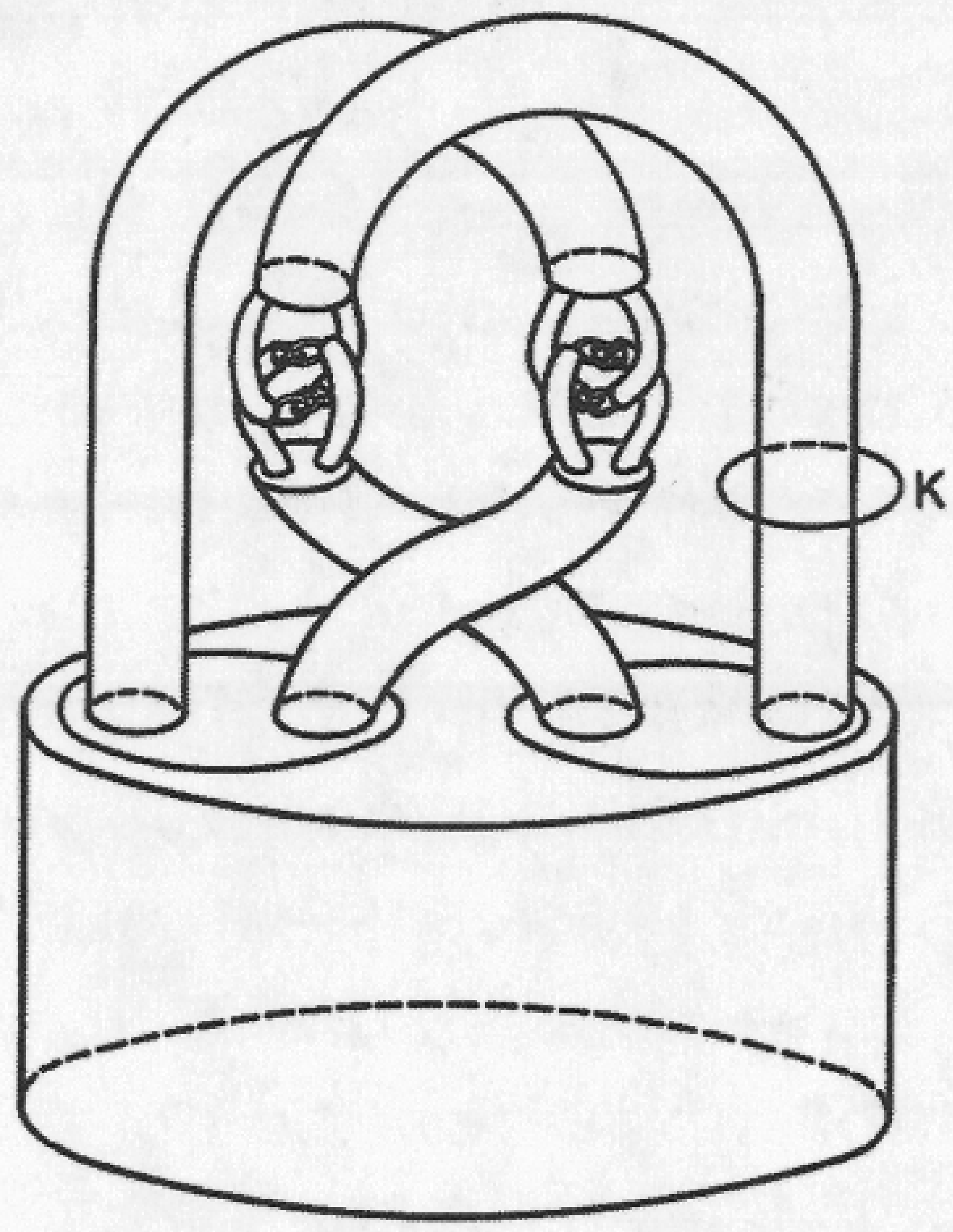
Štan'ko's argument did not apply in all of the places that we would have liked, but we believed that just a little effort would make it work every place. Štan'ko thought so, and he published another short paper to that effect.[29] I was so convinced that it would work in general that I assigned the problem to Ric Ancel, a graduate student. However, Štan'ko's argument had not been carried out carefully enough. Ric worked unsuccessfully for a year to extend the Štan'ko techniques. People all over the United States set about extending his techniques—all without success. I thought the extension should be an easy one. Then I learned that Bob Daverman at Tennessee had tried unsuccessfully, that Bob Edwards at UCLA had tried unsuccessfully, and that Štan'ko himself had tried unsuccessfully. I started feeling guilty. An advisor is supposed to pick solvable problems for his students.

Ric and I decided to work together on the problem. It was a hot, Wisconsin summer. Every day we would sit sweating in our offices and push the problem around for hours at a time. After one and a half months of steady work, we gave up. It was too hard. We could push the difficulty to one side or the other, but always it would reappear someplace else. We decided to work on something else, and we got really involved in the new problem—to such a degree, in fact, that I started to dream about it. One night at 2:00 A.M., my eyes suddenly popped open. I sat up in bed next to Ardyth. I knew how to extend Štan'ko's techniques. I do not know how the answer came to me. I couldn't sleep. I dressed quietly and went walking on the dark streets of Madison. No one was around. I checked the ideas for all of their consequences. I checked for absurdities. I couldn't find any. The picture was wonderful.

Earlier, fixing one difficulty had created new difficulties. The solution was simple: push one point, but in the push don't disturb any of the point's neighbors, and don't tear it way from its neighbors. Yes, it's a simple solution. The only problem is that it is impossible to do. If you don't allow tearing, then moving one point will inevitably move neighboring points. Such an action is simply impossible, unless the point is a really big point! But points have size zero. Now, the real nature of the solution was this: decide which points needed to be moved; increase the size of each point that had to be moved by expanding it into a ball or disk; then, keeping the boundary of the disk fixed so as to not disturb surrounding points, move the inside of the disk. Carry out this process for all of the (infinitely many) points that had to be moved. After this had been done so as to untie all of the infinitely many knots, shrink all of the expanded points back down to real points. The process was exactly the Štan'ko process, except that each step had now been fractured into three substeps, each of which could be performed in a completely controlled way that did not create new difficulties. The next morning, I showed Ric the idea. It was technically difficult to carry out, and it took him a full year of further work to write his thesis.[30]

Bob Edwards also worked on the Štan'ko problem. He had grown up on Long Island and studied at the University of Michigan. He has served for many years as a mathematical ambassador in geometry and topology, taking the news around and helping other people with ideas. After spending months on the Štan'ko work, he found that Štan'ko's argument could be used to understand the exact form of infinite knotting in high dimensions. Bob had the great idea of applying this picture of knotting to the double suspension problem. He found that if one looked "sideways and skeewompus" at the four critical edges of the diamonds making up the new space, they looked a lot like an infinitely knotted set. He could show, he said, that many double suspensions were, in fact, manifolds.

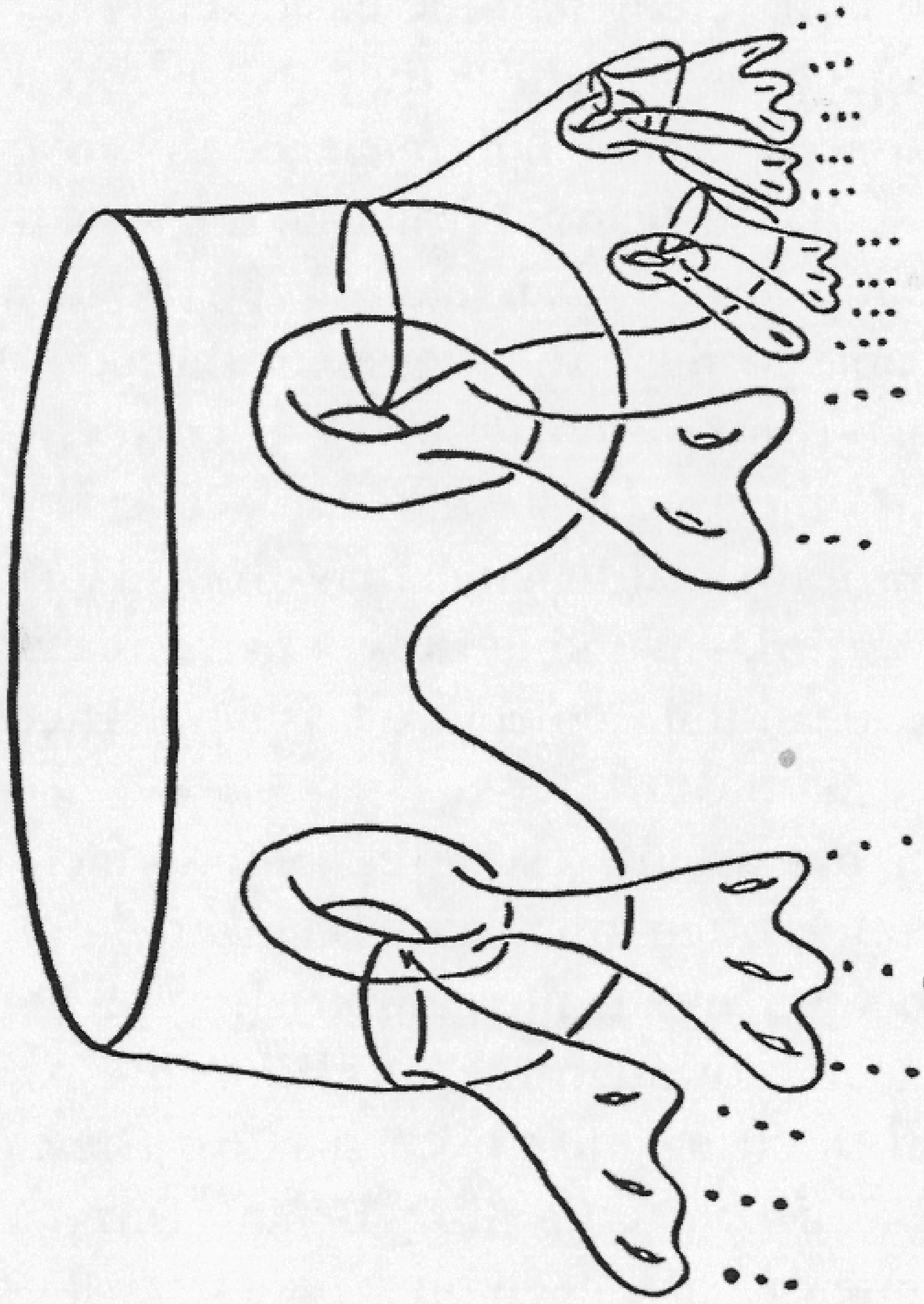
This was an exciting time. Bob's result was incredible. He had spent the previous two years traveling all over the United States picking up the techniques he needed for this problem. My student David Wright had shown me some of Bob's techniques. But there were difficulties. We had no written version of what he had done. We had no idea how he proceeded. He was writing some master work on the thing, but he couldn't prove the general theorem associated with the Milnor double suspension problem. He also couldn't handle some of the most common examples. I had understood



Alexander Horned Sphere

Štan'ko's work, too, and I thought he must use that. Štan'ko seemed to give such a clear view of what must be happening that I believed Bob could finish the problem if he could do one example. Things were becoming clearer and clearer to me, but I needed one more idea. I avoided working hard on the problem, however, for almost two years because I was sure that Bob would finish it off any day. We finally invited Bob to visit us at the University of Wisconsin, and he outlined his insight for us: in high-dimensional spaces all infinite knotting could be traced to ghostly approximations of high-dimensional objects called Alexander Horned Spheres. Bob didn't actually find these horned spheres, but he found something enough like them that he could usually unknot all of the problems.

Bob left, and I thought and thought. I knew what to look for now. I would never have believed that Bob would find something so explicit, but when I looked carefully, I found not just ghostly approximations of these horned spheres, but the horned spheres themselves. I was amazed! The construction of these spheres could be based on a crazy infinite construction. Reach a hand toward the infinite knotting. As the knotting becomes more tightly bound and things get too constricted, split the fingers into multiple tiny fingers. When things get even more constricted, split the tiny fingers into still tinier fingers, ad infinitum. Russ McMillan called the hands with multiple fingers "the grope." (Ric Ancel sent a picture and quotation from Einstein: "How do I work?" asked Einstein. "I grope.") In the limit—in the end—the fingertips nestle onto the knotted structure. Thicken the groping hands. The result is a knotted horned sphere. Now it is time for the old exploding point trick: explode all of the infinitely many fingertips. The knotting



The Grope

disappears like magic. Analyze what has happened, and the proof is finished. The solution sounds easier than it was. I couldn't believe how pretty the picture was. I sat for most of a day just staring at the pictures.

The new results led to important new conjectures, inconceivable earlier: Conjecture A—it is possible to decide whether a space is a manifold by checking a short list of simple conditions. The truth of the conjecture would require the truth of two further conjectures that I simply call Conjectures B and C.

I wrote to Bob, telling him the basic additional ideas and the results and conjectures to which they led. Before I gave my first public talk on my theorems, Bob managed to prove Conjecture B using a key idea that he learned from the doctoral thesis of one of my students, Dan Everett.[31] Dan had learned the idea from me, but only Bob was able to push the argument through. Within a year, Frank Quinn had proved Conjecture C—incorrectly. In this unsatisfactory state, Conjecture C stood for a decade.

Last year, four mathematicians—from Florida, Michigan, Brazil, and New York—working in two pairs, managed to resolve Conjecture C.[32] Beautiful Conjecture C is false. There are infinitely many counter examples.

This story has an interesting epilogue which illustrates the fact that the truth is often more surprising and more beautiful than we imagined beforehand. (An example is Joseph Smith's astonishment upon learning through the revelation the beautiful plan of life recorded in Doctrine and Covenants 76.) The examples found in the refutation of Conjecture C by Bryant, Ferry, Mio, and Weinberger filled more than the need presented by that conjecture. The beautiful algebra developed by the Princeton School had largely mirrored, in algebra, the geometric properties of manifolds. But the algebraic mirror revealed disturbing gaps in the geometric landscape: where algebra suggested the probable existence of special manifolds, those special manifolds occasionally did not, could not exist. The miracle of the Bryant-Ferry-Mio-Weinberger examples of nonmanifolds constructed by the local, infinite processes of the Texas School was that they exactly filled the gaps suggested by the algebraic mirror constructed by the Princeton School. Perhaps these new mathematical models of space, whose very existence

Brazilian Washington Mio with his family. Mio is one of the four mathematicians who determined that Cannon's Conjecture C has many counter examples and is therefore false. 1992. Courtesy Washington Mio.

was unsuspected ten years ago, are in fact as natural and as important as manifolds themselves. At the very least, in a limited way they fuse more perfectly the geometric and algebraic views of our world and fuse more perfectly the work performed by mathematicians with disparate views.

Human Parallels, or Parables: Debt and Appreciation

We turn now to people parallels, or parables. The people we have been discussing have been more to me than mathematicians. They have also been friendly, kind and compassionate, moral and exemplary, loyal and fun, and helpful and supportive when our family was in need. Almost without exception they have had faiths different from ours. To us they have exemplified the best as described by the Savior's parable where people from differing backgrounds were reconciled in a moment of need:

> Who is my neighbor? . . . A certain man went down from Jerusalem to Jericho, and fell among thieves, which stripped him of his raiment, and wounded him, and departed, leaving him half dead. . . . A certain priest . . . when he saw him, passed by on the other side. And likewise a Levite. . . . But a certain Samaritan, . . . when he saw him, he had compassion on him, And went to him, and bound up his

> wounds. . . . Which now of these three, thinkest thou, was neighbor unto him? (Luke 10:29–36)

Ardyth and I have lived approximately twenty-five years away from the centers of the Church. We have during that time incurred a great debt toward many kind Samaritans. The Brigham Young University community must show the same kindness and appreciation toward the strangers in our midst that others showed us when we were strangers. As I talk about mathematicians as people rather than mathematicians, I pay tribute not only to them, but also to those who have committed their lives, as people of a variety of faiths, to our great cause at BYU. I say to them, "We appreciate you. We need you for the excellence of your knowledge and teachings and lives. We need you for what you teach us about life and about ourselves. We need more like you. We know that we do not agree on all things. But we look forward to the time when we both know much more than we know now so that we can see eye to eye."

A number of these mathematicians have taught me by example because they and their families have survived tremendously difficult times with goodness intact. Lipman Bers, George Pólya, and the parents of Ric and Ester Ancel either fled from or survived the horrors of the Nazi concentrations camps.

One account states that "Lipman Bers in 1934, as a young radical had to flee his native Latvia following a fascist coup d'etat. Four years later, having just received his Ph.D. at the University of Prague, he had to flee again, this time because he was a Jew."[33] He is well-known in the mathematical community as an advocate for human rights. Though having suffered the oppression of the Nazis, Bers studied and admired the work of a rabid Nazi youth named Otto Teichmüller, who helped hound Jewish mathematicians at Göttingen, wrote wonderful theorems, volunteered for the Russian Front, and was never heard from again. I have always been touched by Bers's appreciative and sensible response to Teichmüller's work. Bers quotes Plutarch: "It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem."[34]

George Pólya was a Hungarian mathematician I mentioned earlier. Famous for his work in Germany during the twenties and thirties, Jewish Pólya nevertheless had to flee Germany for America during the Nazi regime. As he aged, Pólya turned his great talents to the teaching of teachers. He was greatly beloved. Pólya invited me, as a freshman, to his home, introduced me to his wife, and showed me his mathematical notebooks written over many years. He took time to write me a letter of encouragement during my mission in Austria. He encouraged me to work hard and to do a little mathematics regularly because, as he said in German, "He who rests, rusts."

A number of mathematicians taught me—by little things they did—what I consider important life lessons. When I was just beginning to learn the mathematics that Bill Thurston was studying, Bill and I found ourselves the lone strangers at an extended mathematics conference in Houston. I hardly knew the basic terminology of the subject. Bill suggested a problem. I said that I found the problem really interesting and that I would go off and think about it. Bill said, "Oh, no. It would be a lot more fun to think about it together." And so we did.[35]

One year, Bob Edwards, my friend who worked on the Štan'ko problem, told his department chairman, "I haven't done very well this year. I don't think I deserve a raise." How many of us have that kind of integrity?

Additionally, other mathematicians took us in and cared for us as a family. R. H. Bing, from the Texas school, hired us at Wisconsin. He took our children for rides on the little train at the park. He and Mary Bing lent us their home for two weeks when we didn't have a place to stay. R. H. took the children boating in his speedboat and let them steer, telling them not to mind what their Mom and Dad said about speeding. He saw that we got fellowships, promotions, and enough money to live on. He attended Covenant Presbyterian Church regularly but even then couldn't leave mathematics alone. His daughter, Gay, reported seeing him reach up to write on an imaginary blackboard during the sermon. Later he reached up and erased what he had written.

Ed Burgess was raised in rural Texas and studied topology with R. L. Moore at the University of Texas at Austin. Ed Burgess was my doctoral advisor. He took care of his students. When he decided that I was worth taking a chance on as a sophomore at the University of Utah, he went out and scrounged up a graduate fellowship for me. The dean called me in and explained that, like student athletes, I had no legal obligation to stay at the University of Utah, but that I was morally obligated to pay the fellowship back if I went to another university for graduate school. Ardyth and I used those fellowship moneys to get married after five years of writing letters to each other.

Mary Ellen Rudin[36] was the first person to visit the hospital after our child was born with Down's Syndrome. Having raised such a child herself, she wanted to encourage us. When the child died a year later, she mourned with us.

Further, some mathematicians, dead long before any of us were born, taught me when I was still a high-school student about values and about truth. For example, Karl Friedrich Gauss noted:

> There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science.[37]

My favorite quotation from Henri Poincaré rivals even my favorite from the Book of Mormon, which says, "And if ye will lay hold upon every good thing, and condemn it not, ye certainly will be a child of Christ" (Moro. 7:19). Poincaré talks about truth, its beauty, the reason we love it, the reason we fear it, and its essential unity. We tend to separate sacred and secular truth in a way that I have never been able to understand; I have always believed there is only one truth. Poincaré explains how the two truths, if there are indeed two, cannot be separated. Here is the quotation:

> Truth should not be feared, for it alone is beautiful. When I speak here of truth, assuredly I refer first to scientific truth: but I also mean moral truth. I cannot separate them, and whosoever loves the one cannot help loving the other. These two sorts of truth when discovered give the same joy; each when perceived beams with the same splendor, so that we must see it or close our eyes. In a word, I liken the two truths, because the same reasons make us love them and because the same reasons make us fear them.[38]

Since my childhood, kind Samaritans have cared for my family. The following story is typical of our experiences living in many different neighborhoods. In 1947, I was four years old, the youngest of three small children living with Mother and Father in Bowling Green, Ohio. We were the only Mormon family in town. My father traveled with H. J. Heinz to South Africa on an extended trip to look into the prospects for building a new food-processing plant there. My mother was expecting a fourth child. While my father was gone, my mother suffered a miscarriage. She lost so much blood that the doctor abandoned her to what he termed inevitable death while he cared for accident victims who had just arrived at the hospital. Although Mother did not die, she was very ill for a long time before father returned home. Our neighbors saw to it that Mother was nursed and we children were fed and cared for in every way. We remember those people, Grace Schulz, Ruth Putnam, and others, with great love.

These stories are my own parables because they have taught me, like the parable of the Good Samaritan, that differences do not have to engender fear. Unfortunately, we occasionally have a large streak of fear towards those who are different, toward those who disagree with us—fear that they will corrupt us or cost us our uniqueness. We fear that secular truth will destroy moral truth. But how are we to serve the world if we are afraid of the world, if we are driven by fears that those we disagree with will destroy what is unique about us? I think that we can be unique in the best possible way only if we abandon fear and concentrate on exercising the highest standards in our personal actions and thought.

Here is my personal academic creed:

> I will act with courage and not from fear—fear of what others may expect or think, fear of my own inadequacies.
>
> I will speak freely, openly, publicly. I will remember that our knowledge of truth, even revealed truth, proceeds by approximation according to our ability and experience and that difficult issues can be understood and resolved best in an atmosphere where the evidence—physical, spiritual, or intellectual—can be freely and openly discussed.
>
> I will learn from those who do not agree with me. In particular, I will not impute bad motives to those who do not agree with me. I will instead examine their evidence, their arguments, and their conclusions and weigh each thoughtfully and carefully. I will remember that bad feelings arise when evidence is ignored and people are treated with disrespect and that, since the experiences of individuals vary widely, differences in point of view, even momentarily awkward ones, need to be welcomed, understood, and appreciated.

> I will not fear being wrong, for which of us has not been wrong? I will instead fear being dishonest in what I understand or being unwilling to change as my understanding grows. I will fear using shoddy arguments, presenting weak evidence, and ignoring good evidence and hiding behind cleverness with words or behind the incorrect use of authority. I will not demand that others accept my evidence and arguments but will have faith that good evidence and valid arguments will in the long run prevail.
>
> I will not be embarrassed by those who do what I think is wrong, for they are responsible for their own actions. I will instead try to see the issues clearly and react to the issues directly and honestly.
>
> I will not presume that because I have authority over another (whether an employee, child, or student) that I can demand their loyalty. I will remember that loyalty can be earned but not bought or compelled. I will remember that in academics, as in all of life, any true influence arises not from position, but from good and honest work—persuasion, love, and long-suffering. Influence is not something bestowed by a university or anyone else but comes, when it does, from the good work of the faculty, students, alumni, staff, and administration.

Postscript 1995: Perhaps this creed, my own attempt to implement the principles of Christ's parables, can serve as an approach to the treatment of differences among people. Ardyth and I took at least ten years of married life to learn to handle our differences compassionately and gracefully. I am now trying to learn to handle my differences with people in the larger world.

This talk is, of course, both mathematically and nonmathematically, about the perennial problem of dealing with differences among people. This fundamental and difficult problem is the real subject of the Savior's parable of the good Samaritan. The parable derives its special strength or bite from the fact not that the Samaritan was a good man, but that the good man was a Samaritan, an outsider, a despised and feared person, a stranger, religiously incorrect.

This speech was written in 1993 as we watched BYU struggle with the problem of differences in a self-conscious way. The public dialogue exhibited at least three emotional tendencies, sometimes explicitly, sometimes only implicitly, whose merits, in the best tradition of the great university, deserve to be publicly examined. The first tendency was to fear the strangers of other faiths in our midst as a potential threat to our uniqueness as a church-run university. The second tendency was to assume that all

LDS scholars in our midst, sharing as they do a common label, did, or at least should, share all of the same views. The third tendency, on both sides, was to evaluate difference of view as willful malice. This talk was my attempt to enter the public dialogue in the most gentle and constructive manner I could muster.

The key to overcoming tendency one, fear of strangers, is and always has been one of inversion: imagine ourselves as that stranger in the midst. Because I lived and worked among people of other faiths for many years, this inversion was, for me, an easy matter; the value to me of the stranger's views was a tested and confirmed reality.

Tendency two, judging people by their labels, is a consequence of one's retreat from reasoned dialogue—from replacing reason with sloganeering, labeling, and packaging. We Mormons have differences among ourselves, and we will be in reality strangers to each other until we learn to recognize and treat our differences lovingly and respectfully. The fact that any people share common labels, such as "Mormon" or "Intellectual" or "Conservative" or "Feminist," does not confer upon them a well-defined, common background, experience, knowledge, or views.

Tendency three, the distrust of disagreement, leads to one of the great traps of our public world, the curtailment of public dialogue. Any truly important issue has many aspects and is deserving of open, respectful, and passionate discussion in an atmosphere free of fear. We could do much better in this regard. Essayist Wendell Berry warns of the possibility of forming not only an extreme right and left, but also an extreme middle which "looks upon all critics as traitors," which "equates the government with the country, loyalty to the government with patriotism."[39]

In remembering the lesson of Jesus's great parable, we have the opportunity of avoiding the traps of the extreme middle, loaded labels, and fear of the other. My hope is that we can add the unique qualities of faith and inspiration to the strengths of other great universities without throwing away other qualities as profound.

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NOTES

[1]George Pólya, *How to Solve It: A New Aspect of Mathematical Method* (Princeton, N.J.: Princeton University Press, 1971), 208.

[2]Personal communication with author.

[3]G. H. Hardy, *A Mathematician's Apology* (New York: Cambridge University Press, 1967), 82–85.

[4]Stephen Weinberg, *Dreams of a Final Theory* (New York: Pantheon, 1992), 67.

[5]Constance Reid, *Hilbert* (New York: Springer, 1970), 220.

[6]See R. Courant and D. Hilbert, *Methoden der Mathematischen Physik,* 2 vols. (Berlin: Springer, 1968).

[7]Schrödinger to Wien, February 22, 1926, printed in Jagdish Mehra, *The Historical Development of Quantum Theory,* 5 vols. (New York: Springer, 1987), vol. 5, part 2, p. 583.

[8]Marguerite Kay, *Bruegel* (New York: Paul Hamlyn, 1969).

[9]James Gleick, *Genius: The Life and Science of Richard Feynman* (New York: Pantheon, 1992), 378.

[10]See Donald J. Albers, G. L. Alexanderson, and Constance Reid, *International Mathematical Congresses: An Illustrated History, 1893–1986* (New York: Springer, 1986).

[11]William Dunham, *Journey through Genius: The Great Theorems of Mathematics* (New York: John Wiley and Sons, 1990), 10.

[12]See Tobias Dantzig, *Number: The Language of Science* (Garden City, N.Y.: Doubleday, 1954).

[13]Rudin to J. W. Cannon, January 1993.

[14]See J. W. Cannon, "Mathematics in Marble and Bronze: The Sculpture of Helaman Rolfe Pratt Ferguson," *Mathematical Intelligencer* 13 (1991): 37.

[15]"One Saturday Gerhard Gauss was making out the weekly payroll, . . . unaware that his young son was following the proceedings. Coming to the end of his long computations, Gerhard was startled to hear the little boy pipe up, 'Father, the reckoning is wrong, it should be. . . .' A check of the account showed that the boy [Karl Friedrich Gauss] was correct." E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1937), 221.

[16]See Karl Friedrich Gauss, *General Investigations of Curved Surfaces of 1827 and 1825,* trans. James Caddall Morehead and Adam Miller Hiltebeitel (Princeton: Princeton University Press, 1902).

[17]Bell, *Men of Mathematics,* 500–503.

[18]See Bell, *Men of Mathematics,* ch. 26; Bernhard Riemann, *Collected Works,* ed. Heinrich Weber (New York: Dover, 1953).

[19]Riemann, *Collected Works,* 275.

[20]See H. Poincaré, *Oeuvres,* 6 vols. (Paris: Gauthier-Villars, 1953), 6:183, for the original text, or J. W. Cannon, "The Recognition Problem: What Is a Topological Manifold?" *Bulletin of the American Mathematical Society* 84 (1978): 835, for a rough translation of the complete passage.

[21]See J. W. Milnor, "On the Total Curvature of Knots," *Annals of Math* 52, no. 2 (1950): 248–57.

[22]See the list of all Fields Medalists in Albers, Alexanderson, and Reid, *International Mathematical Congresses.*

[23]See R. H. Bing, *The Collected Papers of R. H. Bing,* ed. Sukhjit Singh, Steve Armentrout, and Robert J. Daverman, 2 vols. (Providence: American Mathematical Society, 1988).

[24]R. Lashof, ed., "Problems in Differential and Algebraic Topology: The Conference in Seattle, 1963," *Annals of Math* 81 (1965): 565–91.

[25]Leslie C. Glaser, "On a Well-known Method for Trying to Obtain a Noncombinatorial Triangulation of S^n for $n \geq 5$," *Duke Mathematics Journal* 35 (1968): 147–54.

[26]Leslie C. Glaser, "On Double Suspensions of Arbitrary Nonsimply Connected Homology *n*-spheres," in *Topology of Manifolds,* ed. James C. Cantrell and C. H. Edwards Jr. (Chicago: Markham, 1970), 5–17.

[27]See M. A. Štan'ko, "Approximation of Imbeddings of Compacta in Codimensions Greater Than Two," *Dokl. Adad. Nauk SSSR* 198 (1971): 783–86 or in Societ Math. Dokl. 12 (1971): 906–9; and M. A. Štan'ko, "Approximation of Compacta in E^n in Codimensions Greater Than Two," *Mat. Sbornik* 90, no. 132 (1973): 625–36 or in Math USSR Sbornik 19 (1973): 615–26.

[28]Štan'ko, "Approximation of Imbeddings of Compacta," 783–86.

[29]M. A. Štan'ko, "Approximation of Imbeddings of Manifolds in Codimension One," *Mat. Sbornik* 23 (1974): 456–66.

[30]See F. D. Ancel and J. W. Cannon, "The Locally Flat Approximation of Cell-like Embedding Relations," *Annals of Mathematics* 109 (1979): 61–86.

[31]See Robert D. Edwards, "The Topology of Manifolds and Cell-like Maps," in *Proceedings of the International Congress of Mathematicians, Helsinki, 1978* (Hungary: Academia Scientiarum Fennica, 1980), 111–27.

[32]See J. L. Bryant and others, "Topology of Homology Manifolds," *Bulletin of the American Mathematical Society* 28 (April 1993): 324–28.

[33]Donald J. Albers, Gerald L. Alexanderson, and Constance Reid, *More Mathematical People: Contemporary Conversations* (New York: Harcourt Brace Jovanovich, 1990), 3.

[34]Lipman Bers, "Quasiconformal Mappings and Teichmüller's Theorum," in *Analytic Functions,* ed. R. Nevannlina and others (Princeton, N.J.: Princeton University Press, 1960), 90.

[35]See Albers, Alexanderson, and Reid, *More Mathematical People,* 324–43.

[36]See Albers, Alexanderson, and Reid, *More Mathematical People,* 283–303.

[37]Bell, *Men of Mathematics,* 204.

[38]See Henri Poincaré, *The Value of Science* (New York: Dover, 1958), 11–12, for the entire quotation.

[39]See Wendell Berry, *Recollected Essays, 1965–1980* (San Francisco: North Point Press, 1981), 154.