Courtesy Wulf Barsch

Detail, *I Am a Brother to Dragons and a Companion to Owls,* by Wulf Barsch. With the firmament often acting as his subject, Barsch's paintings invoke a sense of the infinite through mathematical symbols. See *To Depict Infinity: The Artwork of Wulf Barsch* on pages 33–38.

To Journey Beyond Infinity

Kent A. Bessey

The notion of infinity has fascinated philosophers, scientists, and mathematicians for millennia. Its enigmatic nature seemed to thwart all attempts to unlock its secrets. Scriptural allusions to the infinite evoke a similar sense of mystery. Few have been as intrigued by the concept of infinity—or as tenacious in trying to understand it—as the German mathematician Georg Cantor. Between 1874 and 1884, Cantor published numerous papers that illuminated some of the shadowy regions of the infinite.[1] He discovered a remarkable realm where half of a pie is as large as the whole, infinity comes in different sizes, and miracles are mathematically plausible.

Cantor's journey to infinity began in a rather pedestrian way as he sought to determine whether two collections of things—called sets—contain the same number of objects. This could be accomplished, of course, simply by counting the objects in each set and comparing the results. If the sets are large enough, however, one may not finish counting within a lifetime, many lifetimes, or ever. Another approach, which lies at the core of Cantor's reasoning, is to find a way (if possible) of pairing the objects in each set.[2] For example, suppose we want to know if BYU's Marriott Center can accommodate all individuals intent on attending a devotional address. Let P represent the set of people who would like to be seated in the Marriott Center. Let S represent the set of available seats. Inviting all such individuals into the auditorium and asking them to take a seat will quickly reveal the relative sizes of the sets P (people) and S (seats). If every seat is taken and no one is left standing, then the number of individuals is the same as the number of seats. In this case, we say the sets P and S have the same *cardinality.*[3] After all the seats are taken, if there are still people standing, then the number of individuals is more than the number of seats. In this case, we say the cardinality of the set P is greater than the cardinality of the set S. Finally, if everyone is able to find a seat and there are still seats available, then we say the cardinality of the set P is less than the cardinality of the set S.

Some might wonder how this simple exercise could possibly shed light on the concept of infinity. But consider this: did we at any point need to know the actual number of individuals or seats when comparing the sizes of the sets P and S? No, we did not. The pairing of a person with an available seat circumvented the need to know how many seats there were or the number of attendees at the devotional. Through pairing, we can determine whether the cardinality of a set is less than, more than, or the same as the cardinality of another set without knowing the number of objects in either set.

Armed with this technique, Cantor made discoveries that profoundly altered mathematicians' views of infinity. Applying the pairing procedure to truly large sets—infinite sets such as the set of positive integers {1, 2, 3, 4, . . . }—he concluded that the set of positive even integers {2, 4, 6, 8, . . . } has the same *number* of numbers as the set of positive integers. This follows because a pairing can be made between all the numbers in both sets. More specifically, by doubling each number in {1, 2, 3, 4, . . . } we get a number in {2, 4, 6, 8, . . . }, so that 1 is paired with 2, 2 is paired with 4, 3 is paired with 6, as shown:

1, 2, 3, 4, . . .
2, 4, 6, 8, . . .

Observe also that every number in either set has a partner assigned through this pairing process. Therefore, the cardinality of the set of positive integers is the same as the cardinality of the set of positive even integers.

It is possible to have missed the significance of what we just did. We have shown that a *part* can be as large as the *whole*.[4] In other words, half of a pie can be as large as the whole pie. In the realm of the infinite, ordinary intuition proves inadequate. My students soon discover this when we study infinite sets. As a point of discussion, I have my class read Doctrine and Covenants 84:38, which states: "And he that receiveth my Father receiveth my Father's kingdom; therefore all that my Father hath shall be given unto him." Giving away all that one has is a sure road to penury. Because of this, some have argued that the above scripture refers to the sharing of power. But receiving another's kingdom and being given all that the other has seem more tangible than the sharing of power alone. Suppose this scripture also refers to the sharing of tangible, though perhaps celestial, possessions. Can we still make sense of the scripture? I believe we can. Each semester I assign my students (enrolled in *Foundations of Mathematics*) the following problem:

> Consider the set of positive integers {1, 2, 3, 4, . . .}. Let each positive integer represent a one-ounce gold coin. Suppose all these coins are yours to share. By means of either a formula or a diagram show that you can give an *infinite* number of these coins to an *infinite* number of people while retaining an *infinite* number of coins for yourself.

Faced with a challenging mathematical problem, a person would do well to tackle easier versions of the problem first. In the present context, a good place to start is to identify a way of sharing infinitely many of these gold coins with one other person while retaining infinitely many coins for yourself. A simple solution is to give the other person the gold coins associated with the positive *even* integers while retaining the coins that correspond to the positive *odd* integers. Next, you might try devising a procedure that assigns an infinite number of these coins to an arbitrary but finite number of people while retaining infinitely many coins for yourself. Once you have accomplished this, you are only one step away from solving the original problem. Pictorially, the final step to the solution is elegant and accessible. Arrange the positive integers, which represent the one-ounce gold coins, into an infinite diagram (as depicted below). Observe the pattern of consecutive integers along each diagonal line:

You retain →	1	3	6	10	15	21	. . .
You give →	2	5	9	14	20	. . .	
You give →	4	8	13	19	. . .		
" "	7	12	18	. . .			
" "	11	17	. . .				
" "	16	. . .					
	. . .						
	. .						
	.						

Retain the gold coins corresponding to the first row of this diagram: 1, 3, 6, 10, 15, 21, . . .; give the second person the coins corresponding to the second row: 2, 5, 9, 14, 20, . . .; give the third person the coins corresponding to the third row: 4, 8, 13, 19, . . .; and so forth. In this way, you can share an infinite number of these coins with an infinite number of people while retaining an infinite number of coins for yourself. So indeed, it is possible for an individual in possession of sufficient (that is, infinite) wealth to share an equal portion with others while retaining the original amount of wealth.

Cantor did not stop his investigation of infinity here. What he did next was so breathtaking that prominent mathematicians of his day refused to give him audience to justify his results. He demonstrated that not all infinite sets are of the same size—of the same cardinality—and that some infinite sets are *tremendously* larger than others. He proved, in other words, that there are different sizes of infinity. Most mathematicians were unprepared for such a conclusion, and many dismissed it as fantasy. Today, in contrast, Cantor's work is considered to be among the greatest performed in the field of mathematics. Yet his discovery of higher orders of infinity was more serendipitous than intentional. The whole notion, however, became plausible when he considered the collection of all subsets of a set, where a subset consists of some, none, or all of the objects in the set.

The collection of *all* subsets of a set is called the *power set*. Informally, to obtain a power set we look for all combinations of objects from the given set. For example, the power set of {1, 2, 3} is {Ø, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}, where Ø represents the empty set. If the idea of a power set intrigues you, then I encourage you to take (or audit) an introductory course in set theory. In that course we prove that the cardinality of the power set is always greater than the cardinality of the original set. For instance, the cardinality of the power set of {1, 2, 3} is 8, since it contains 8 subsets, as listed above, while the original set {1, 2, 3} has cardinality 3. More generally, the cardinality of the power set for a set of size n is 2^n, because there are 2^n subsets that can be formed from a set containing n objects.[5] Thus the power set for a set of three objects contains 2^3 subsets. The power set for a set of four objects contains 2^4 subsets, and so forth.

Let us explore how quickly the size of the power set grows as n increases. We have already shown that the power set of {1, 2, 3} contains 2^3 or 8 subsets. Using the general formula, we conclude that the power set of {1, 2, 3, 4} contains 2^4 or 16 subsets; the power set of {1, 2, 3, 4, . . . , 10} contains 2^{10} or 1,024 subsets; and the power set of {1, 2, 3, 4, . . . , 100} contains 2^{100} or 1,267,650,600,228,229,401,496,703,205,376 subsets, which is more than one million trillion trillion. Exploring further, we discover that the power set of {1, 2, 3, 4, . . . , 1000} contains 2^{1000} subsets, where 2^{1000} equals a number that has 302 digits! Thus, by means of the power set, any finite set can be used as a stepping stone to build another, much larger, finite set.

Notwithstanding the impressive size of the power set for large finite sets, it was not until Cantor turned his attention to forming the power set of the *infinite* set of positive integers {1, 2, 3, 4, . . .} that the unfathomable occurred—this power set landed him on the other side of infinity. As with finite sets, the cardinality of the power set of {1, 2, 3, 4, . . .} is *larger* than the cardinality of the set itself. How can two infinite sets be of different sizes? How would we prove that they are? If we can show that *every* pairing between these sets leaves out objects from one of the sets, then the set with unpaired objects is larger than the other. Such is the case with {1, 2, 3, 4, . . .} and its power set. Every pairing between the sets leaves objects in the power set without partners.[6] Therefore, the power set of {1, 2, 3, 4, . . .} is a larger infinite set than {1, 2, 3, 4, . . .}! For this reason, the set of positive integers is referred to as *countably* infinite and its power set as *uncountably* infinite, or more succinctly, *uncountable*.[7]

The power set of the positive integers is so large that few mathematicians would claim to have a genuine sense of its size. I wonder whether Moses 1:35 is relevant here: "For behold, there are many worlds that have passed away by the word of my power. And there are many that now stand, and *innumerable* are they unto men; but all things are *numbered* unto me, for they are mine and I know them" (emphasis added). Also, I suspect the reference to "many" in John 14:2, "In my Father's house are many mansions," is a spectacular understatement.

We can now restate our wealth-sharing problem with greater precision:

> Consider the set of positive integers {1, 2, 3, 4, . . .}. Let each positive integer represent a one-ounce gold coin. Suppose all these coins are yours to share. By means of either a formula or a diagram show that you can give a *countably* infinite number of these coins to a *countably* infinite number of people while retaining a *countably* infinite number of coins for yourself.

The adventurous reader might wonder if the above problem is still possible if we replace "the set of positive integers" with "the *power set* of the positive integers" and "countably infinite" with "uncountable":

> Consider the *power set* of the positive integers. Let each subset of the positive integers represent a one-ounce gold coin. Suppose all these coins are yours to share. By means of either a formula or a diagram show that you can give an *uncountable* number of these coins to an *uncountable* number of people while retaining an *uncountable* number of coins for yourself.

The answer is yes. The problem can be solved, and the proof is within the grasp of a third-year mathematics student. Another question some might have is whether there is an infinite set whose cardinality is *greater* than that of the set of positive integers but *less* than that of the power set of the positive integers. The answer: "We don't know." This question is undecidable using the axioms of set theory. Most mathematicians, however, do not believe such an infinite set exists.[8]

Have we reached the end of our journey in search of larger infinite sets? We have scarcely begun. For now we have a method, or some might say a metaphor, that inexorably churns out larger and larger sets of larger and larger "infinities." If the cardinality of the power set of the positive integers is beyond human comprehension, then what about the *power set of the power set* of the positive integers? Ineffably large! Dare we ask: what about the *power set of the power set of the power set* of the positive integers? The mathematically minded should be overcome by cerebral exhaustion. Power set upon power set upon power set *ad infinitum* gives new meaning to the scripture "Be still and know that I am God" (D&C 101:16).

Although some may doubt the reality of infinite sets of different sizes, I am confident that the reader has experience with two particular infinite sets—one larger than the other. The first is the set of positive integers, which is countably infinite. The other is the set of decimal numbers, called real numbers. The set of real numbers has the same cardinality as the power set of {1, 2, 3, 4, . . .}, which is uncountable. Their equivalence can be established, as always, by appropriately pairing real numbers with subsets of {1, 2, 3, 4, . . .}. Real numbers are often represented by points (locations) on a line—called the real number line—where zero lies in the middle, negative numbers to the left, and positive numbers to the right. This line is a geometric realization of an uncountable set. The real numbers saturate the number line in the sense that any finite segment of the line contains uncountably many points. As an aside regarding Amulek's description of the Atonement as "an infinite and eternal sacrifice" (Alma 34:10), the saturating quality of the real numbers suggests the possibility of compressing an eternity of experiences into a finite amount of time.[9]

Germane to our discussion of infinite sets is a result Cantor proved regarding the real number line and its higher-dimensional analogs. One dimension is characterized by restricted movement along a line—what we might call forward/backward movement. Two dimensions enjoys a greater degree of freedom, characterized by forward/backward and left/right movement. Three dimensions is characterized by forward/backward, left/right, and up/down freedom of movement. Some may question my use of the word movement, since motion requires an additional time dimension. But I have chosen this word solely for its intuitive appeal, which helps convey the desired sense of spatial dimension.

Maintaining his remarkable record for disquieting the mathematical community, Cantor proved that one-dimensional space (a line) has *exactly* the same number of points as two-dimensional space. Stated more precisely, the cardinality of the set of points that form a line is the same as the cardinality of the set of points that form two-dimensional space.[10] This should be upsetting. On a piece of paper, draw a line of whatever length you would like. The line you have drawn is made up of exactly the same number of points as the points that make up the entire piece of paper! In other words, you can take an edge of the piece of paper and rearrange the points on that edge to form an entirely new sheet of paper (albeit, an extremely flat one) of whatever length and width you choose.

But if one- and two-dimensional spaces have the same number of points, then perhaps two- and three-dimensional spaces also have the same number of points. Such is the case. Consequently, one-dimensional space and three-dimensional space have the same number of points! Now, find a box. Identify an edge on the box (not a side, but an edge). You can rearrange the points on that edge to form an entirely new box of whatever size you would like. If you want to get more imaginative, the points on that edge can be rearranged to form *any* three-dimensional object whatsoever. Suddenly, the "feeding of the five thousand" becomes mathematically conceivable: a few morsels can be rearranged point-for-point to fill baskets full of food![11]

Equally tantalizing is the relationship between two and three dimensions. Imagine, if you can, what it would be like living in a two-dimensional world, where spatially there is only forward/backward and left/right.[12] There is no up/down freedom of movement—it just does not exist for you. Your world is restricted to a flat surface, such as a tabletop. You cannot jump *up* off the table because in your world there is no up. Geometrically, you have a length and a width but no height. For the sake of simplicity, suppose you are a circle with its interior as the inside of your body. Outside the circle is the world around you in this two-dimensional universe. Unknown to you, there is a larger three-dimensional world, of which your world comprises two of the dimensions. Consequently, there is an up/down direction but only for those living in three dimensions. Such individuals would have full view of you and others on the tabletop, but they could easily remain out of sight from everyone in your two-dimensional world. Yet denizens of three dimensions could choose to be seen, at least in part, by coming in contact with the tabletop. For example, imagine that a glass (which is three-dimensional) is placed on the table. The bottom of the glass, being in contact with the tabletop, can be viewed by those living on the surface of the table. It would seem to have appeared out of nowhere and can disappear just as quickly when the glass is lifted off the table.

Perhaps more intriguing, suppose you, as a circle and its interior, become ill with cancer. You opt for surgery to remove a tumor. Removing the tumor in your two-dimensional world requires cutting through your circumference (the outer portion of the circle that encloses your interior body). This is the only way a doctor in your world could surgically get to the tumor. But suppose a benevolent, competent individual in the three-dimensional world wanted to help. The tumor is in full view and is easily accessible in three dimensions without needing to cut *through* you. Assistance could come from above as the three-dimensional benefactor performed surgery *from inside you*, and the entire procedure would be hidden from the eyes of two-dimensional observers.

Analogously, the fourth or a higher dimension might be a place for spirits or heavenly beings. They would have an unobstructed view of our three-dimensional world while remaining invisible to us. They could make contact with us and even change the course of events. Mathematicians are not perturbed by the notion of higher-dimensional spaces. In fact, from a mathematical standpoint the fourth dimension is quite prosaic. Cantor commonly worked with *n*-dimensional space, where *n* can be any positive integer. For example, in 53-dimensional space, instead of having just forward/backward, left/right, and up/down, you have 50 additional choices of directions.[13] Talk about freedom! But even 53 dimensions would seem restrictive compared to still higher-dimensional spaces. Freest of all, one may surmise, is a *countably infinite* dimensional space, where you have a countably infinite number of directions to choose from. Yet even this space pales by comparison to an *uncountable* dimensional space, where you have an uncountable number of degrees of freedom when moving about, revealing worlds within worlds within worlds.

The enchanting qualities and protean nature of infinity continue to captivate and stir the imagination. Cantor transformed the mathematical landscape by his inquiries into the infinite. He discovered a realm of paradox and poetry of a sort never before encountered, where human intuition has little authority. He demonstrated the value of a single, simple, right idea. Above all, he altered mathematicians' view of infinity as an interminable process (a verb) to an actual entity (a noun). It was as though he had been inspired by the imagery evoked in William Blake's poem "Auguries of Innocence": "[To] hold infinity in the palm of your hand."

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1. Cantor's six-part treatise on set theory appears in the German journal *Mathematische Annalen*.

2. In mathematical terminology, a pairing of the objects in two sets is referred to as either a "one-to-one correspondence" or a "bijection."

3. Cardinality refers to the number of elements in a set; conceptually, the set's magnitude or size.

4. Using the most fundamental of all measures in Measure Theory—the counting measure—yields the same measurement for the set of positive integers as it does for the set of positive even integers. Hence, we may say, "a *part* can be as large as the *whole*."

5. This follows from the Fundamental Counting Principle.

6. For the intrepid reader, I give an informal proof that the power set of {1, 2, 3, 4, . . .} is larger than {1, 2, 3, 4, . . .} in cardinality. Before doing so, it is customary (for the sake of notational simplicity) to identify the power set of {1, 2, 3, 4, . . .} with the collection of infinite strings—called sequences—of zeros and ones. We can represent any subset of {1, 2, 3, 4, . . .} by a sequence of zeros and ones. A "one" in the sequence denotes the presence of a number in the subset and a "zero" denotes its absence. For example, the subset {1, 2, 3} is represented by 1, 1, 1, 0, . . . (zeros thereafter); the subset {1, 3, 4} is represented by 1, 0, 1, 1, 0, . . . (zeros thereafter); and the subset {2, 4, 5, 7} is represented by 0, 1, 0, 1, 1, 0, 1, 0, . . . (zeros thereafter). The process also works in reverse: given any sequence of zeros and ones we can reproduce the subset of {1, 2, 3, 4, . . .} that corresponds to it. To prove that the power set of {1, 2, 3, 4, . . .} is larger than {1, 2, 3, 4, . . .} in cardinality, we need only show that *every* pairing between the positive integers and sequences of zeros and ones will leave some sequence without a partner. Given *any* pairing between the positive integers and sequences of zeros and ones, we can construct a sequence that has no partner. To illustrate this, consider a *particular* pairing between the positive integers and sequences of zeros and ones:

1	is paired with	1, 0, 1, 1, 0, 0, . . .
2	is paired with	0, 0, 1, 0, 1, 1, . . .
3	is paired with	1, 1, 0, 1, 1, 0, . . .
4	is paired with	0, 1, 0, 1, 1, 1, . . .
.	.	.
.	.	.
.	.	.

We can construct a sequence of zeros and ones that is different from the sequences listed above. For instance, consider the sequence 0, 1, 1, 0, . . . ; it differs from the first sequence in its first digit; it differs from the second sequence in its second digit; it differs from the third sequence in its third digit, and so on (see bolded digits in the diagram below). This means the sequence 0, 1, 1, 0, . . . is not the same as any of the sequences listed; hence, it is not paired with any positive integer.

1	is paired with	**1**, 0, 1, 1, 0, 0, . . .
2	is paired with	0, **0**, 1, 0, 1, 1, . . .
3	is paired with	1, 1, **0**, 1, 1, 0, . . .
4	is paired with	0, 1, 0, **1**, 1, 1, . . .
.	.	.
.	.	.
.	.	.

No matter what pairing we are given between the positive integers and sequences of zeros and ones, we can always find sequences without partners. Therefore, the set of sequences of zeros and ones is *larger* in cardinality than the set of positive integers. Consequently, the power set of {1, 2, 3, 4, . . .} is larger than {1, 2, 3, 4, . . .} in cardinality.

7. In mathematical parlance, *countably infinite* refers to a set that has the same cardinality as the set of positive integers {1, 2, 3, 4, . . .}. Whereas a *countable* set can be either a finite set or a countably infinite set. For this reason, we do not shorten countably infinite to countable. There is no such ambiguity in the uncountable case.

8. The common belief among mathematicians that there is no infinite set whose cardinality is greater than that of the positive integers but less than that of the power set of the positive integers is formalized in The Continuum Hypothesis.

9. Assuming time is sufficiently divisible.

10. (For advanced mathematics readers only.) Let R represent the set of real numbers. To prove that a line and two-dimensional space have the same cardinality as sets of points, it suffices to show that there is a bijection from R onto the Cartesian product RxR. This can be accomplished by "unzipping" the real numbers. To illustrate this idea, consider the decimal expansion for a real number between 0 and 999: $d_1\, d_2\, d_3\, .\, d_4\, d_5\, d_6\, d_7\, d_8\, d_9\, d_{10}\, d_{11} \ldots$, where d_j is a digit between 0 and 9. Assign this real number to $(d_2\, .\, d_4\, d_6\, d_8\, d_{10} \ldots,\ d_1\, d_3\, .\, d_5\, d_7\, d_9\, d_{11} \ldots)$ in the Cartesian product. Extending this assignment to all of R, while handling non-unique expansions with care, gives a bijection from R onto RxR. Hence, a line and two-dimensional space have the same cardinality as sets of points.

11. Assuming matter is sufficiently divisible.

12. In 1884, Edwin A. Abbott wrote a delightful story about life in a two-dimensional world. His book *Flatland* is still in print, published by Dover Publications.

13. Orthogonal (perpendicular) directions.